Statistical properties of a discrete version of the Ornstein-Uhlenbeck process

Hernán Larralde
Centro de Ciencias Físicas, UNAM Apdo. Postal 48-3, C.P. 62251, Cuernavaca, Morelos, Mexico
(Received 1 September 2003; published 20 February 2004)

A discrete version of the Ornstein-Uhlenbeck process is discussed which arises from a simple generalization of the master equation of the random walk. The calculation of the statistical properties of the free propagator for this process can be obtained using essentially the same formalism as for simple random walks. These calculations are carried out in some detail for the one-dimensional case. The usual equation for the evolution of the probability distribution of the Ornstein-Uhlenbeck process is recovered in the continuum limit if the jump distribution has a finite variance. However, the discrete process is also well defined for long tailed jump distributions and, thus, can be used to describe a Lèvy walk under the effect of a harmonic potential. Finally, a brief discussion of the generalization of this process to describe random walks in general potentials is presented and briefly compared with results arising from the fractional diffusion approach.

DOI: 10.1103/PhysRevE.69.027102 PACS number(s): 02.50.-r, 05.40.Fb

I. INTRODUCTION

Brownian motion and its discrete counterpart random walks have found innumerable applications in physics, chemistry, economics, biology, and many other areas of research. The same is true of the Ornstein-Uhlenbeck (OU) process [1–4], which is the most generalized continuous stationary Gaussian Markov process [2], and can be thought of physically as an overdamped Brownian particle in a harmonic potential. However, there appears to be relatively little discussion on the discrete counterparts of this process. Most attempts have been made by considering discrete random walks on a lattice and introducing the effect of the potential as a spatially varying bias [3]. More recently, this approach has been generalized to include the possibility of long range jumps as a basis for the derivation of appropriate generalizations to the Fokker-Planck equation (see, for example, Refs. [5,6], and references therein). This work presents a different approach in which the main effect of the potential is to relocate the center of the jump distribution. For the harmonic potential this gives rise to a simple generalization of the (continuous space) random walk which is a discrete time analog of the OU process. The characteristic functions and statistical properties of this discrete OU process can be computed in essentially the same manner as for normal random walks [7]. As occurs in the continuous OU process, the discrete process tends to a stationary distribution and the continuous OU process is shown to be recovered in a particular scaling limit. However, the discrete OU process is also well defined for step distributions which possess no moments and the limiting stationary distribution is not Gaussian in this case. These can be interpreted as Lèvy walks on a harmonic potential, and the results agree with those obtained from a fractional diffusion approach [8]. Finally, a generalization to nonharmonic potentials is briefly discussed.

II. DISCRETE OU PROCESS

The master equation describing the evolution of the discrete OU process is given by the following simple generalization of the master equation for a random walk [7]:

$$P_{n+1}(x)=\int_{-\infty}^{\infty}\phi(x-\gamma y)P_n(y)dy, \tag{1}$$

where γ is a constant and $\phi(j)$ can be interpreted as a "jump distribution" which will be assumed to have zero mean without loss of generality.

A particle transport interpretation of the process described by Eq. (1) is that it represents a random walk which takes place on a harmonic potential. The potential acts during the time interval between jumps by changing the landing position y of the nth step to γy, the initial position of the $(n+1)$th step, much like a flea on an appropriate slippery bowl that jumps to a certain position, then slides down the side of the bowl before jumping again, and so on. If the flea's motion while sliding down the bowl is assumed to be overdamped, then γy is the result of sliding down a harmonic potential for a fixed time interval τ. Clearly, the overdamped equation of motion is

$$\dot{y}=-ky \quad \text{thus} \quad y(t+\tau)=e^{-k\tau}y\equiv\gamma y. \tag{2}$$

This interpretation is appealingly simple, and opens the possibility of contemplating the effects of more complicated potentials which act on the particle during the intervals between steps; an example of such extensions will be discussed further on. From this point of view, the distribution $P_n(x)$ is the probability density of the flea's landing sites. The actual probability density of the position of the flea as a function of time must be computed together with the sliding motion; this will not be pursued in this work. However, we can also define $Q_n(y)$ as the probability density for the jump departure sites. Clearly, if x_n is the arrival site of the nth jump, the departure site of the nth+1 jump will be $y_{n+1}=\gamma x_n$. Thus $Q_{n+1}(y)$ can be computed directly from $P_n(x)$ by a simple rescaling of the variable. This is not the case for more complicated potentials, for which the statistics of the landing sites and of the departure sites can be qualitatively different.

Presently we will be concerned with the process as described by Eq. (1). A further discussion of the difference between landing and departure sites will be conducted in reference to extensions to other potentials.

Clearly, for the process under consideration, very different behaviors will arise depending on whether $|\gamma|$ is greater or smaller than 1: if $|\gamma|>1|$ the process diverges exponentially, while if $|\gamma|<1$ the process is, in some sense, confined and becomes stationary. Of course, when γ is exactly equal to one, the usual random walk is recovered.

The continuous OU process is easily recovered from Eq. (1) by writing time as $t=n\tau$, taking $\gamma\sim 1-k\tau$, and expanding $\phi(x-\gamma y)=\phi(x-y+[1-\gamma]y)\approx\phi(x-y)+\phi'(x-y)k\tau y+\cdots$. The right hand side of Eq. (1) then reads

$$\int_{-\infty}^{\infty}(\phi(x-y)+\phi'(x-y)k\tau y+\cdots)P_n(y)dy. \quad (3)$$

Introducing the change of variable $j=x-y$ and integrating by parts gives

$$\int_{-\infty}^{\infty}\phi(j)\left(1-k\tau\frac{\partial}{\partial j}[x-j]+\cdots\right)P_n(x-j)dj$$
$$=\int_{-\infty}^{\infty}\phi(j)P_n(x-j)dj$$
$$+k\tau\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\phi(j)[x-j]P_n(x-j)dj+\cdots. \quad (4)$$

Finally, assuming the distribution $\phi(j)$ to be sharply peaked at the origin, with a second moment $\sigma^2\sim 2D\tau$, then, to order τ, one obtains

$$P(x,t)+\frac{\partial P(x,t)}{\partial t}\tau+\cdots=P(x,t)+D\tau\frac{\partial^2}{\partial x^2}P(x,t)$$
$$+k\tau\frac{\partial}{\partial x}xP(x,t)+\cdots, \quad (5)$$

where $P_n(x)$ has been written as an explicit function of time and the fact that $\phi(j)$ has zero mean has been used. Cancelling the $P(x,t)$ on both sides and dividing by τ yields the evolution equation of the OU process.

However, Eq. (1) is a simple enough generalization of the random walk, that it can be dealt with by the same techniques developed for that process. A Fourier transform casts it into the recursion relation

$$\hat{P}_{n+1}(\theta)=\hat{\phi}(\theta)\hat{P}_n(\gamma\theta), \quad (6)$$

with the initial condition $\hat{P}_0(\theta)=e^{i\theta x_o}$. The solution to this recursion relation is easily seen to be

$$\hat{P}_n(\theta)=e^{i\gamma^n\theta x_o}\prod_{m=0}^{n-1}\hat{\phi}(\gamma^m\theta),\quad n=1,2,3,\ldots. \quad (7)$$

This expression allows a direct calculation of the cumulants of $\hat{P}_n(\theta)$ in terms of those of the step distribution function, if these exist. That is, if $\hat{\kappa}_n(\theta)$ is the cumulant generating function associated with $P_n(x)$ and $\hat{q}(\theta)$ that associated with $\phi(j)$, then

$$\hat{\kappa}_n(\theta)\equiv\sum_{m=0}^{\infty}i^m\frac{\kappa_n(m)}{m!}\theta^m$$
$$=i\gamma^n\theta x_o+\sum_{\mu=0}^{n-1}\sum_{m=0}^{\infty}i^m\frac{q(m)}{m!}\gamma^\mu\theta^m. \quad (8)$$

Thus, having assumed that $\phi(j)$ has zero mean, we have

$$\kappa_n(1)=\gamma^n x_o\quad\text{for}\quad m=1,$$

$$\kappa_n(m)=\frac{1-\gamma^{mn}}{1-\gamma^m}q(m)\quad\text{for}\quad m>1. \quad (9)$$

If $|\gamma|>1$, the above expressions do not converge as $n\to\infty$. However, if $|\gamma|<1$, then as $n\to\infty$, the above expressions do converge, and $\hat{P}_n(\theta)$ tends to the Fourier transform of the stationary distribution, namely,

$$\hat{P}_\infty(\theta)=\prod_{m=0}^{\infty}\hat{\phi}(\gamma^m\theta). \quad (10)$$

In what follows, only values of γ in the interval [0,1] will be considered. It should be noted that even in this case, careless choices of $\phi(x)$ can lead to extremely singular distributions $P_\infty(x)$. For example, if $\phi(\theta)=\cos\theta$ and $\gamma=1/2$, the stationary distribution is the uniform distribution in the interval $[-1,1]$, however, if $\gamma<1/2$ the stationary distribution has support on a Cantor set [2], with fractal dimension $d_f=\ln(1/2)/\ln(\gamma)$. This amusing behavior is a consequence of the fact that $\phi(x)$ is itself singular; however it does suggest that no limit theorem is generally applicable in the large n limit.

On the other hand, when $\gamma\to 1^-$, we can again write $\gamma=1-k\tau$ and $m=t'/\tau$, so that $\gamma^m=(1-k\tau)^{t'/\tau}\to e^{-kt'}$ as $\tau\to 0$. If we further assume that the characteristic function of the jump distribution can be written as $\hat{\phi}(\theta)\sim 1-\Gamma\tau|\theta|^\alpha+o(\tau)$ when $\tau\to 0$, where $0<\alpha\leqslant 2$ [7], then

$$\prod_{m=0}^{n-1}\hat{\phi}(\gamma^m\theta)\sim\exp\left(-\Gamma|\theta|^\alpha\int_0^t e^{-\alpha kt'}dt'\right). \quad (11)$$

Thus, in this particular scaling limit, we do reach a stable distribution [2,7], whose characteristic function is given by

$$\hat{P}(\theta,t)\sim\exp\left[ie^{-kt}\theta x_o-\Gamma|\theta|^\alpha\left(\frac{1-e^{-\alpha kt}}{\alpha k}\right)\right]. \quad (12)$$

If the second moment of the step distribution is finite, then $\alpha=2$ and the characteristic function of the OU process is again recovered [4].

It should be remarked that the asymptotic expression (12) coincides with the results derived from the fractional diffusion approach to Levy flights in harmonic potentials [8]. Other statistical properties can also be obtained as extensions of the results for random walks. For example, the statistics of first entrance into an interval for this process. Also, higher dimensional versions of this process can give rise to a rich behavior. These courses of inquiry will be pursued elsewhere.

III. NONHARMONIC POTENTIALS

We now discuss how to generalize the approach to the case of nonharmonic potentials. The idea simply consists of substituting the sliding stage of the process, in the sliding flea interpretation, to the overdamped motion of a particle in an arbitrary potential $V(x)$. Given an initial point y and a time interval τ, the departure point will be given by a $x=f(y,\tau)$, solution to the overdamped equation of motion $\dot{x}=-V'(x)$. The recursion relation for the distribution of arrival sites then reads

$$P_{n+1}(x)=\int_{-\infty}^{\infty}\phi(x-f(y,\tau))P_n(y)dy. \tag{13}$$

The continuum limit attained by taking $\tau\to 0$ must now be taken carefully. The usual approach would be to write $f(y,\tau)\approx y-V'(y)\tau$ and then to expand $\phi(x-y+V'(y)\tau)\approx\phi(x-y)+\phi'(x-y)V'(y)\tau$, as before, leading to

$$P_{n+1}(x)\approx\int_{-\infty}^{\infty}\phi(x-y)P_n(y)dy + \tau\int_{-\infty}^{\infty}\phi'(x-y)V'(y)P_n(y)dy. \tag{14}$$

However, there is no guarantee that the second integral on the right hand side converges. It is easy to see that $P_n(x)$ is at least as "wide" as $\phi(x)$ for any potential: the most confining potential would be one which brings the particle to the origin, say, after each step, and in this case $P_n(x)=\phi(x)$; less confining potentials allow the departure sites to be distributed around the origin, leading to a landing site distribution which is wider than $\phi(x)$. Thus, the convergence of the integral is especially troublesome in processes in which the step distribution has long tails. On the other hand, if the jump distribution is sufficiently localized, the integral does converge, and the usual scaling arguments can be applied leading to the appropriate Fokker-Planck equation.

An example that illustrates the problems of convergence in Eq. (14), is the case of Lèvy walks in potentials of the form $V(y)=(k/\beta)|y|^{\beta}$ with $\beta\geqslant 2$, for which some beautiful results were obtained in Refs. [9,10] from a fractional diffusion approach. Though the problem cannot be solved entirely, what can be shown is that for any finite τ there are striking differences between the statistics of the landing sites and the statistics of the departure sites.

As the system is symmetric we consider the motion on the positive side. The equation of sliding motion will be

$$\dot{x}=-kx^{\beta-1}. \tag{15}$$

Thus, if we denote the arrival site by x and the site of departure by z for the next jump after a time interval τ, these are related by

$$z=\frac{x}{(1+(\beta-2)x^{\beta-2}k\tau)^{1/(\beta-2)}}\equiv f(x,\tau). \tag{16}$$

It is worth emphasizing that as x tends to infinity, $z\to 1/[(\beta-2)k\tau]^{1/(\beta-2)}$, and, thus, the support of the distribution of the departure sites is always bounded. Thus, for any choice of $\phi(x)$, all the moments of the distribution of departure sites are finite. This is in contrast with the distribution of the arrival sites which may not have even have finite second moment, which would be the case if the jump distribution corresponds to that of a Lèvy walk: following the same lines of the argument given above, should $\phi(x)\sim 1/|x|^{1+\alpha}$ for large $|x|$, then also $P_n(x)\sim 1/|x|^{1+\alpha}$ in this limit (since the support of the departure sites is bounded). Thus, both the distribution of departure sites and of landing sites differ strongly from each other and from those obtained in fractional diffusion in steep potentials [9,10]. The reason for this discrepancy comes not only from the possible divergence of the integral in Eq. (14), but also from the fact that the approximation $f(y,\tau)\approx y-ky|y|^{\beta-2}\tau$ only holds over the region $|y|\ll(k\tau)^{1/(\beta-2)}$ in the integral of Eq. (13), and the remaining terms do not necessarily vanish fast enough to yield a fractional Fokker-Planck equation as a consistent continuous limit.

In summary, we have presented a simple generalization of the master equation for random walks which gives rise to a discrete Ornstein-Uhlenbeck process. The generalized master equation can be analyzed using the same techniques as those developed for normal random walks. If the second moment of the jump distribution is finite, the continuum limit of the discrete process recovers the continuous Ornstein-Uhlenbeck process; however, the discrete process is well defined also for long tailed jump distributions, and the continuum limit in this case coincides with the results obtained from the fractional diffusion approach. The approach presented in this work provides a simple physical picture for random walks in the presence of a potential, which can be easily generalized to other potentials. However, we have shown that the similarities with the results obtained from the fractional diffusion approach do not hold in steeper potentials.

Further generalizations to higher dimensions and to distributed sliding times may also be interesting. But at least the former represents a rather difficult mathematical task.

ACKNOWLEDGMENTS

Partial financial support by CONACYT through Grant No. 32173, and DGAPA-UNAM through Grant No. IN 112200 is gratefully acknowledged.

[1] G.E. Uhlenbeck and L.S. Ornstein; Phys. Rev. **36**, 823 (1930); and in *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover, New York, 1954).
[2] W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1958), Vol. 2.
[3] M. Kac, Am. Math. Monthly **54**, 369 (1947); and in *Selected Papers on Noise and Stochastic Processes* (Ref. [1]).
[4] H. Risken, *The Fokker-Planck Equation*, 2nd ed. (Springer-Verlag, Berlin, 1989).
[5] R. Metzler, E. Barkai, and J. Klafter, Europhys. Lett. **46**, 431 (1999).
[6] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[7] G. H. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, Amsterdam, 1994).
[8] S. Jespersen, R. Metzler, and H.C. Fogedby, Phys. Rev. E **59**, 2736 (1999).
[9] A.V. Chechkin, J. Klafter, V.Yu. Gonchar, R. Metzler, and L.V. Tanatarov, Phys. Rev. E **67**, 010102(R) (2003).
[10] A.V. Chechkin, V.Yu. Gonchar, J. Klafter, R. Metzler, and L.V. Tanatarov, eprint cond-mat/0306601.